The Dimension of Signed Graph Valid Drawing

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Abstract

A signed graph is a graph with a sign assignment to their edges. Drawing a signed graph in $\mathbb{R}^k$ means finding an embedding of the set of nodes into $\mathbb{R}^k$ such that, for each node, all its positive neighbors (friends) are closer than its negative neighbors (enemies). This work addresses the problem of finding $L(n)$, the smallest dimension k such that *any* graph on n nodes has a valid drawing in $\mathbb{R}^k$, with respect to euclidean distance. We show that $L(n) = n - 2$ by demonstrating that any graph on n nodes can be embedded in $\mathbb{R}^{n-2}$ and that there exists a signed graph on n nodes that does not have a valid drawing in $\mathbb{R}^{n-3}$.

1 Introduction

The problem of drawing signed graphs has received increasing attention in recent years due to its different applications in social networks, such as in opinion formation [7], consensus decision-making [1], the evolution of beliefs [9], and community detection [2]. A signed graph is an undirected graph where each edge has an associated sign, positive or negative. In [5], the definition of a *valid drawing* for signed graphs was introduced. A drawing is said to be valid if for every node its positive neighbors are closer than its negative neighbors with respect to the Euclidean distance. In the same work, a full characterization of the set of complete signed graphs with a valid drawing in the real line was given. This characterization was proven to be testable in polynomial time. However, for general graphs, it was shown that deciding whether or not a graph has a valid drawing in the real line is an NP-complete problem [3]. This result implies that finding the smallest k such that a *given* signed graph has a valid drawing in $\mathbb{R}^k$ is an NP-Hard problem.

The following question remains open: What is the smallest k, called $L(n)$, such that *any* signed graph with n nodes has a valid drawing in $\mathbb{R}^k$? In this work we show that $L(n) = n - 2$. In order to prove $L(n) \leq n - 2$, we use results from distance geometry. Furthermore, to show that $L(n) > n - 3$ we provide a construction of a signed graph that was developed specially for this result. This proves that $L(n) = n - 2$.

This work is organized as follows. In section 2, we provide definitions that will be used throughout this work. Section 3 presents related work. In section 4, we present the proof that $L(n) \leq n - 2$, and we show in section 5 that $L(n) > n - 3$. Finally, in section 6, we present our conclusions.

2 Signed Graphs and Valid Drawings

Next, we formally define signed graphs and valid drawings. We consider only finite graphs with no parallel edges and no self-loops. A *signed graph* is defined as follows:

Definition 1. *A* signed graph *is a graph $G = (V, E)$ together with a sign assignment $f : E \to \{-1, +1\}$ to their edges.*

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Equivalently, a signed graph can be defined as a graph G together with a bipartition of the set of edges E. The set of edges E is partitioned into

$$E^+ = \{e \in E : f(e) = +1\}$$

and

$$E^- = \{e \in E : f(e) = -1\}.$$

We use $G = (V, E^+ \cup E^-)$ to denote a signed graph composed vertices V, and edges $E = E^+ \cup E^-$.

Given a signed graph $G = (V, E^+ \cup E^-)$, we define *friends* and *enemies* for each vertex in G. Let i be a vertex, and $N_i = \{j \in V : (i, j) \in E\}$ be the set of neighbors of i. Let us define the set of *friends*, or *positive neighbors*, of a vertex i as the set

$$N_i^+ := \{j \in N_i : (i, j) \in E^+\}.$$

Equivalently, let us define the set of *enemies*, or *negative neighbors*, of vertex i as the set

$$N_i^- := \{j \in N_i : (i, j) \in E^-\}.$$

Let $G = (V, E^+ \cup E^-)$ be a signed graph. Let $D : V \to \mathbb{R}^k$ be an embedding of the set of vertices of G into $\mathbb{R}^k$. We call D a drawing of G in $\mathbb{R}^k$. Moreover, we define the validity of a drawing as follows:

Definition 2. *Let $G = (V, E^+ \cup E^-)$ be a signed graph, and let D be a drawing of G in $\mathbb{R}^k$. We say that D is a* valid drawing *of G if:*

$$\forall i \in V \quad \forall j^+ \in N_i^+ \quad \textit{and} \quad \forall j^- \in N_i^- \quad \|D(i) - D(j^+)\|_2 < \|D(i) - D(j^-)\|_2 \tag{1}$$

Definition 2 captures the requirement that every node is closer to its friends than to its enemies. In the case that there exists a valid drawing of a given signed graph G in $\mathbb{R}^k$, we say that G *has a valid drawing* in $\mathbb{R}^k$. Otherwise, we simply say that G is a signed graph without a valid drawing in $\mathbb{R}^k$.

We use the notation $d(x, y) = \|x - y\|_2$ to denote the euclidean distance between $x, y \in \mathbb{R}^k$.

3 Related Work

Several researchers have designed algorithms to provide good signed graph embedding algorithms in $\mathbb{R}^2$ [12, 10, 11]. These algorithms do not necessarily satisfy the condition required for a drawing to be valid, since there are known signed graphs without a valid drawing in $\mathbb{R}^2$ [5].

For the real line, the set of complete signed graphs with a valid drawing was characterized in [5]. Then, in [3], a new characterization of the set of complete graphs with a valid drawing on the real line was given: Let G be a complete signed graph, then G has a valid drawing if and only if its positive subgraph is a proper interval graph. Therefore, deciding the existence of such an embedding can be done in polynomial time by considering the positive subgraph of the signed graph. For general graphs, not necessarily complete, it was proven that deciding whether such an embedding exists or not is an NP-complete problem[3]. This result implies that finding the smallest k such that a given signed graph has a valid drawing in $\mathbb{R}^k$ is also an NP-Hard problem.

An optimization version of the embedding problem on the real line has been studied [8], Here the goal is to minimize the number of restrictions (1) that are broken. It was shown that when the graph is complete, local minima for this problem coincide with local minima of the Quadratic Assignment Problem.

4 Upper Bound on L(n)

In order to prove that $L(n) \leq n - 2$, we need to show that any signed graph on n nodes can be embedded into a Euclidean space of dimension $n-2$. For this, we use a previous result stated in the context of Distance Geometry [6]. An important problem in Distance Geometry is to find a set of points in an Euclidean space whose pairwise distance are equal to some given distances.

In [4], the authors consider the question of finding the smallest $\lambda(n)$ such that every collection of $\binom{n+1}{2}$ lengths satisfying $\lambda(n) \leq \ell_{ij} \leq 1$ can be realized as the edge lengths of an simplex on $n+1$ points $p_1, \ldots, p_{n+1}$

in $\mathbb{R}^n$. That is, $\ell_{ij} = d(p_i, p_j)$. For example, $\lambda(2) = \frac{1}{2}$. In Theorem 2, they derive an explicit formula for $\lambda(n)$, which is approximately $1 - \frac{1}{n}$. Given a signed graph G on $n+1$ nodes and a sufficiently small ϵ, we can require that every node is *exactly* at a distance of $1-\epsilon$ from its friends and 1 from its enemies. This set of edge lengths forms a simplex which can be embedded in $\mathbb{R}^n$. This shows that any signed graph on $n+1$ points can be embedded into $\mathbb{R}^n$, so $L(n) \leq n-1$.

Using ideas presented in [4], we go a step further and show that any graph on $n+2$ nodes can be embedded in $\mathbb{R}^n$. Thus, showing that:

Theorem 1. *Let $G = (V, E^+ \cup E^-)$ be a signed graph such that $|V| = n+2$. Then G has a valid drawing in $\mathbb{R}^n$.*

Proof. In [4], the authors define a square matrix of real numbers L as *allowable* if $\ell_{ii} = 0$ and $\ell_{ij} = \ell_{ji} \geq 0$. They define $\mathcal{L}_n$ as the set of all such matrices of dimension $n+1$. They say that $L \in \mathcal{L}_n$ is *realizable* if there are points $p_1, p_2, \ldots, p_{n+1} \in \mathbb{R}^n$ such that $\ell_{ij} = d(p_i, p_j)$ (in section 2 of [4]). Then, in Lemma 1, see section 4 of [4], the authors show that the set of realizable matrices with an n-dimensional realization form an open subset of $\mathcal{L}_n$ with respect to the distance metric

$$\|L - L'\|_\infty = \max_{1 \leq i,j \leq n+1} |\ell_{ij} - \ell'_{ij}|.$$

In fact, the authors prove that the *degenerate* matrices in $\mathcal{L}_n$, matrices where the realization is smaller than n-dimensional, form the common boundary between the *non-degenerate* and *non-realizable* matrices.

Consider the following set of $n+2$ points: the first n points are at

$$p_i = \frac{1}{\sqrt{2}} e_i = (0, \ldots, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0),$$

where e_i is the i^{th} unit vector. These points are all at a unit distance from each other. The next two points, p_{n+1} and p_{n+2} are located at the two distinct points which are unit distance from all of the first n points. They are in the span of $e_1 + e_2 + \ldots + e_n$. In fact, they are at

$$p_{n+1}, p_{n+2} = \frac{1 \pm \sqrt{1+n}}{n\sqrt{2}}(1, \ldots, 1).$$

Essentially, our construction is two unit simplexes in $\mathbb{R}^n$ "glued together" at the n points that form their base. Now, these $n+2$ points have the property:

$$\forall i < j, \quad d(p_i, p_j) = \begin{cases} 1 & i < n+1 \\ \sqrt{2\frac{n+1}{n}} & i = n+1 \end{cases}.$$

Let the set of distances derived from this realization be expressed in the matrix L_0. We know that L_0 is realizable and non-degenerate (since we derived L_0 from a realization), so if we perturb some of the distances by a sufficiently small ϵ the resulting distances are still realizable.

To apply this to the problem at hand, consider a signed graph G on $n+2$ nodes. Assume that at least one edge, e, is negative. Otherwise, the existence of an embedding is trivial since any embedding will satisfy condition (1). Map the nodes adjacent to e to p_{n+1} and p_{n+2}. Map the rest of the nodes to any of the p_i, for $1 \leq i \leq n$. We slightly adjust the distances. We want every positive edge to have distance $1-\epsilon$ and every negative edge except e to have distance 1. The edge e corresponds to a distance which is more than $\sqrt{2}$, so every negative edge corresponds to a larger distance than every positive edge. By construction, this drawing of G is valid. By the argument in the previous paragraph, this drawing of G is realizable in $\mathbb{R}^n$. $\square$

In conclusion, any graph on $n+2$ nodes can be drawn in $\mathbb{R}^n$. Equivalently, $L(n) \leq n-2$.

5 Lower Bound on L(n)

Previously we showed that the $L(n) \leq n-2$. To proceed, we show that $L(n) > n-3$ and thus $L(n) = n-2$ by showing that there exists a signed graph on n nodes that cannot be embedded into $\mathbb{R}^{n-3}$.

Theorem 2. *There exists a signed graph on $n+3$ nodes without a valid drawing in $\mathbb{R}^n$.*

Construction: We construct the following graph with the $n+3$ nodes: let the set of nodes be

$$V = \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, w_3\}.$$

Let the first n nodes be enemies. That is, v_i and v_j are connected via a negative edge for $i \neq j$. Let the last 3 nodes be enemies. That is, w_i and w_j are connected via a negative edge for $i \neq j$. Let the v_i and w_j be friends for all v_i and w_j. That is, every pair of nodes v_i, w_j is connected via a positive edge. We will assume that this graph on $n+3$ can be embedded in n dimensions and arrive at a contradiction.

Let H_v be a hyperplane of dimension $n-1$ which passes through all the v_i. Note that this hyperplane is not necessarily unique. Assume, by pigeonhole principle and without loss of generality, that w_1 and w_2 are on the same side of H_v, or lie on H_v, and assume that w_1 is at least as far from the hyperplane as w_2. Let H_{w_1} be the hyperplane that is parallel to H_v and that passes through w_1. Ultimately, we will show that condition (1) is violated on w_1. Hence, the drawing cannot be valid, which is a contradiction.

For each $i \in \{1, 2, \ldots, n\}$, let r_i be the shortest distance from v_i to v_j for $j \neq i$:

$$r_i = \min_{j \neq i}\{d(v_i, v_j)\}.$$

By construction, r_i is the distance from v_i to its nearest enemy, so w_1 and w_2 must be within r_i of v_i. Hence, for all i the following inequalities hold:

$$d(v_i, w_1) < r_i \qquad\qquad d(v_i, w_2) < r_i.$$

Let the region F be the intersection of the d balls of radius r_i each centered at their v_i, for $i \in \{1, \ldots, n\}$, between the hyperplanes H_v and H_{w_1}. Both w_1 and w_2 are in F.

Let ρ_i be the distance from w_1 to v_i, for $i \in \{1, 2, \ldots, n\}$: $\rho_i = d(v_i, w_1)$. And let ρ be the maximum of those distances:

$$\rho = \max_i \rho_i.$$

Since w_1 has a friend at a distance ρ away, the distance between w_1 and w_2 must be more than ρ. An example of this construction and notation for $n = 2$ is given in Figure 1.

We will also use the following additional definitions:

- In general, let C_p^r be a ball of radius r centered at the point p. The dimension of the ball will be clear from context.
- Let $\pi(w_1)$ be the projection of w_1 onto H_v.
- Let $R = \max_i d(\pi(w_1), v_i)$ be the distance from $\pi(w_i)$ to the furthest v_i.
- Let F_v be the intersection of the region F and H_v.
- Let $C_{\pi(w_1)}^R$ be the intersection of $C_{w_1}^\rho$ and H_v, where the value of R is implied by the construction.

To show that the signed graph on $n+3$ as described cannot be embedded into $\mathbb{R}^n$, we need to show that it is impossible to place w_2 outside $C_{w_1}^\rho$ but within F. It is sufficient to prove that $F \subseteq C_{w_1}^\rho$ to arrive at a contradiction. We prove that $F \subseteq C_{w_1}^\rho$ with the theory developed in the next paragraphs.

In fact, Lemma 1 shows that we only need to consider hyperplane H_v:

Lemma 1.

$$F_v \subseteq C_{\pi(w_1)}^R \implies F \subseteq C_{w_1}^\rho.$$

Proof. F is defined in terms of balls with centers in H_v. For any z in the segment of the line between v and w_1, consider a hyperplane H_z that is parallel to H_v and that passes through z. As we move z from v to w_1, the intersection of F with H_z shrinks while the intersection of H_z and $C_{w_1}^\rho$ grows. Hence, if $F \cap H_z \subseteq H_z \cap C_{w_1}^\rho$, then, for any z' closer to w_1 in the segment of the line between v and w_1, it holds $F \cap H_{z'} \subseteq H_{z'} \cap C_{w_1}^\rho$. Which, in the case $z = v$, is equivalent to saying $F_v \subseteq C_{\pi(w_1)}^R \implies F \subseteq C_{w_1}^\rho$. □

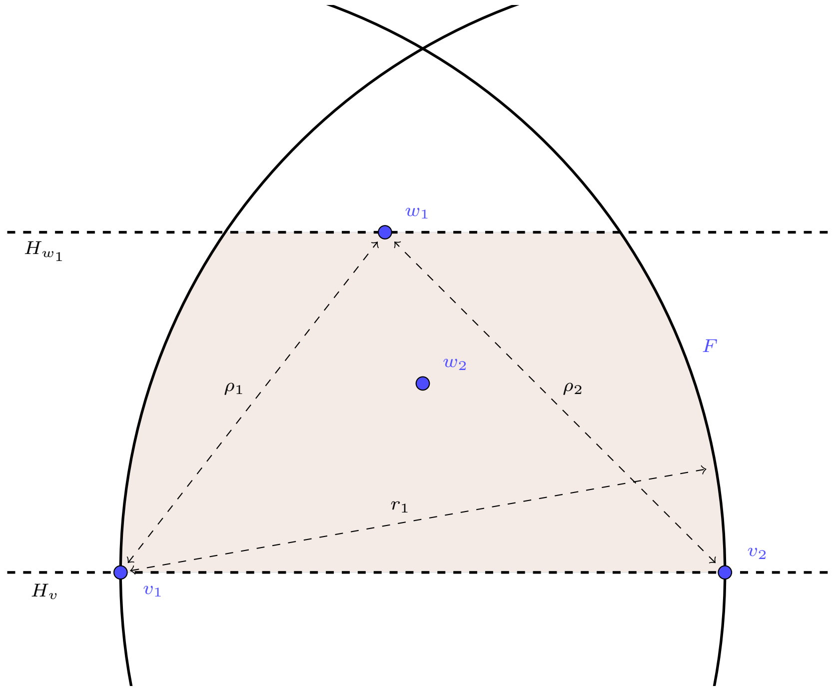

Figure 1: Example of the construction for $n = 2$ in two dimensions. This drawing is not valid since the distance between w_1 and w_2 is less than $\rho = \max_i \rho_i$.

Spherical Caps: From now on, we will be working strictly inside H_v (recall Lemma 1).

For our proof, we need to characterize the region F_v. In particular, part of F_v will lie inside the convex hull of the v_i and part of it will lie outside the convex hull. Since all the v_i are within R of $\pi(w_1)$, it follows that the convex hull of the v_i are inside $C^R_{\pi(w_1)}$. The challenging part of the proof is considering the parts of the region F_v which are outside the convex hull of the v_i.

To characterize this region, we define a *spherical cap*. Let C^r_p be an arbitrary ball and let H be an arbitrary hyperplane. Then we define the spherical cap as:

$$S^r_{p,H} = C^r_p \cap H^+.$$

Unless otherwise specified, we assume that p is on the negative side of the hyperplane: $p \in H^-$. Thus, $S^r_{p,H}$ is the part of the ball C^r_p bounded away from p by H.

We define an important set of up to n spherical caps (one for each v_i). Let $H_{\{\sim i\}}$ be a hyperplane of dimension $n-2$ that passes through the v_j, where $j \neq i$ (it is dimension $n-2$ since we are in H_v). The important spherical caps are:

$$S^*_i = S^{r_i}_{v_i, H_{\{\sim i\}}} = C^{r_i}_{v_i} \cap H^+_{\{\sim i\}}.$$

In words, S^*_i is created by taking the ball centered at v_i with radius r_i and intersecting it with the half-space defined by $H_{\{\sim i\}}$. An example of the spherical caps and the projection for $n = 3$ is shown in Figure 2.

Lemma 2. *Every point in F_v either lies within the convex hull of the v_i or in one of the spherical caps S^*_i for some i.*

Proof. Every point in F_v lies in all the balls $C^{r_i}_{v_i}$ centered at v_i with radius r_i. If it does not lie in any of the spherical caps, then it is within all the facets that defines the convex hull of the v_i, which are the hyperplanes $H_{\{\sim i\}}$. □

In order to prove that the spherical caps S^*_i are also inside $C^R_{\pi(w_1)}$, we rely on the next two lemmas.

Lemma 3 (Spherical Cap Containment)**.** *Let $C^{r_1}_{p_1}$ and $C^{r_2}_{p_2}$ be balls in $\mathbb{R}^k$. Let H be a hyperplane. If the following three conditions hold*

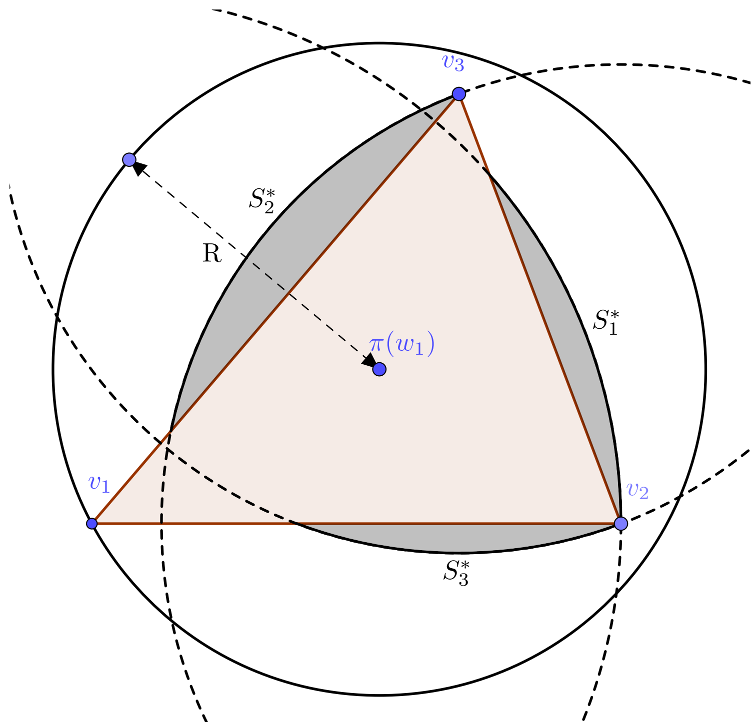

Figure 2: Projection of a drawing for $n = 3$ onto H_v. F_v is the intersection of the dashed circles. F_v is contained in the union of the spherical caps S_i (drawn in gray) and the convex hull of the v_i (drawn in red).

- $p_1, p_2 \in H^-$,
- $r_1 \leq r_2$,
- $H \cap C_{p_2}^{r_2} \subseteq H \cap C_{p_1}^{r_1}$,

then the spherical cap $S_{p_1,H}^{r_1} = C_{p_1}^{r_1} \cap H^+$ *contains the spherical cap* $S_{p_2,H}^{r_2} = C_{p_2}^{r_2} \cap H^+$.

Proof. The argument is essentially a "scaling" and a "translation." If we were to grow the radius of r_1 to r_2 while keeping its intersection with H the same, the spherical cap would shrink. Let $C_{p_3}^{r_2}$ be the ball such that $H \cap C_{p_1}^{r_1} = H \cap C_{p_3}^{r_2}$ and $p_3 \in H^-$. Then,

$$S_{p_3,H}^{r_2} \subseteq S_{p_1,H}^{r_1}.$$

This is the "scaling" step. Now, $C_{p_2}^{r_2}$ is a translation of $C_{p_3}^{r_2}$, where $H \cap C_{p_2}^{r_2} \subseteq H \cap C_{p_1}^{r_1} = H \cap C_{p_3}^{r_2}$. Therefore,

$$S_{p_2,H}^{r_2} \subseteq S_{p_3,H}^{r_2}.$$

Combining the two inequalities gives the desired result. □

Intersections and Containment: Fix an arbitrary node i. We will argue with respect to v_i and its spherical cap S_i^*. Recall that the spherical cap S_i^* is defined in terms of the ball $C_{v_i}^{r_i}$ and the hyperplane $H_{\{\sim i\}}$:

$$S_i^* = S_{v_i,H_{\{\sim i\}}}^{r_i}.$$

Let S_i' be the spherical cap defined in terms of the ball $C_{\pi(w_1)}^R$ and the same hyperplane $H_{\{\sim i\}}$:

$$S_i' = S_{\pi(w_1),H_{\{\sim i\}}}^R.$$

Lemma 4. $S_i^* \subseteq S_i'$

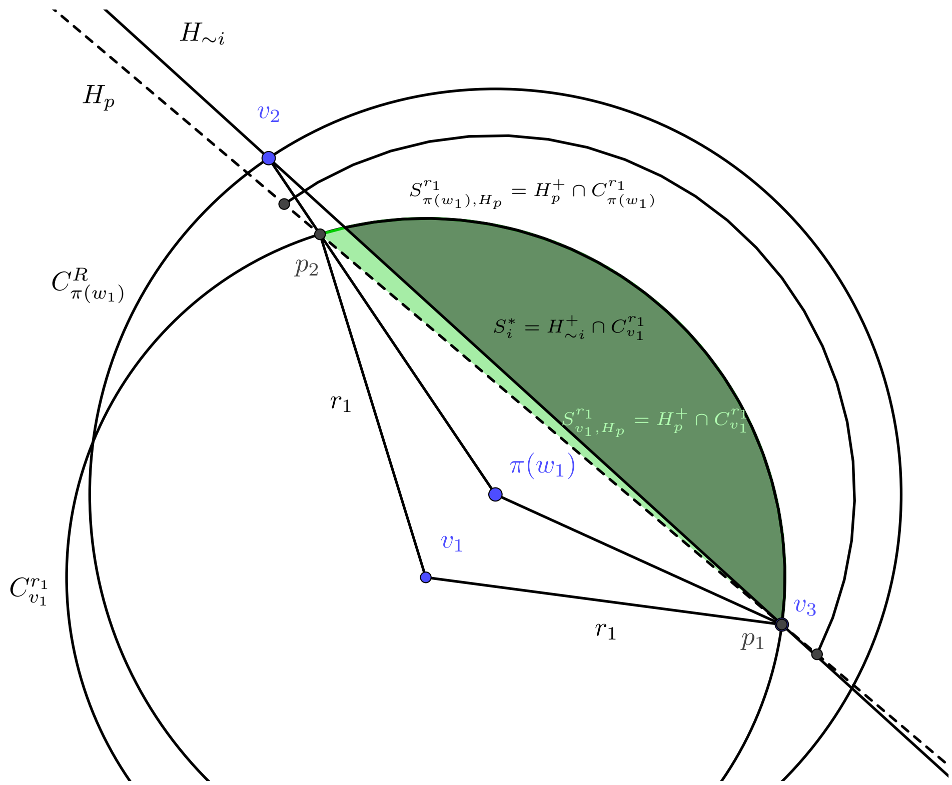

Figure 3: Construction for Lemma 4 when $R > r_1$.

Proof. We can assume that $\pi(w_1) \in H^-_{\{\sim i\}}$. If $\pi(w_1) \in H^+_{\{\sim i\}}$, then we can translate the hyperplane $H_{\{\sim i\}}$ until it passes through $\pi(w_1)$. During this translation, the intersection $H_{\{\sim i\}} \cap C^{r_i}_{v_i}$ shrinks while the intersection $H_{\{\sim i\}} \cap C^R_{\pi(w_1)}$ grows. This is the same logic as Lemma 1.

The proof is presented in two cases: $R \le r_i$ and $R > r_i$.

Case 1. $R \le r_i$

This is an application of Lemma 3.

Recall, from our construction that $d(v_i, v_j) \ge r_i$ for all $j \ne i$ and $d(\pi(w_1), v_j) \le R$. That is, $C^R_{\pi(w_1)}$ contains all the v_j, while $C^{r_i}_{v_i}$ is at most tangent at the v_j. Thus,

$$H_{\{\sim i\}} \cap C^{r_i}_{v_i} \subseteq H_{\{\sim i\}} \cap C^R_{\pi(w_1)}.$$

In addition, $R \le r_i$, so the conditions of Lemma 3 are satisfied.

Case 2. $R > r_i$

Unfortunately, we cannot use the exact same argument as before, since $R \le r_i$ was key to applying Lemma 3.

For all the $j \ne i$, w_1 is a friend of v_j and v_i is an enemy of v_j, so

$$d(\pi(w_1), v_j) < d(v_i, v_j).$$

Let p_j be the point where the ball of radius r_i centered at v_i intersects the line segment from $\pi(w_1)$ to v_j. This point must exist, since $\pi(w_1)$ is inside the ball and v_j is outside. An illustration for this construction for $n = 3$ is shown in Figure 3.

We consider the value of $d(\pi(w_1), p_j)$. We would like to show that $d(p_j, \pi(w_1)) < r_i$. By the triangle inequality, it holds that

$$d(v_i, v_j) \le d(v_i, p_j) + d(p_j, v_j).$$

By construction of p_i, $d(v_i) \leq r_i$ and thus

$$d(v_i, v_j) \leq r_i + d(p_j, v_j).$$

Since friends must be closer than enemies, this implies that

$$d(\pi(w_i), v_j) < r_i + d(p_j, v_j). \tag{2}$$

Finally, by construction, the distance from $\pi(w_1)$ to v_j can be broken down as the sum of the respective distances to p_j

$$d(\pi(w_1), v_j) = d(\pi(w_1), p_j) + d(p_j, v_j).$$

Combining this observation with equation (2), we obtain the required statement

$$d(\pi(w_1), p_j) < r_i.$$

This means that the ball centered at $\pi(w_1)$ with radius r_i contains all the p_j. If we let H_p be the hyperplane through the p_j, then we can apply Lemma 3:

$$S^{r_i}_{v_i,H_p} \subseteq S^{r_i}_{\pi(w_1),H_p}.$$

Since $\pi(w_1)$ is on the same side of $H_{\{\sim i\}}$ as v_i, the p_j are also on the same side, so

$$S^*_i \subseteq S^{r_i}_{v_i,H_p} \subseteq S^{r_i}_{\pi(w_1),H_p} \subseteq C^{r_i}_{\pi(w_1)} \subseteq C^R_{\pi(w_1)}.$$

Finally, we intersect S^*_i and $C^R_{\pi(w_1)}$ with $H^+_{\{\sim i\}}$ to complete the proof,

$$S^*_i = H^+_{\{\sim i\}} \cap S^*_i \subseteq H^+_{\{\sim i\}} \cap C^R_{\pi(w_1)} = S'_i.$$

□

Proof. (Proof of Theorem 2) First, notice that it is sufficient to prove that $F \subseteq C^\rho_{w_1}$, since then w_2 cannot also lie within F and so we have a contradiction.

Lemma 1 says that, in order to prove $F \subseteq C^\rho_{w_1}$ it is sufficient to prove $F_v \subseteq C^R_{\pi(w_1)}$. Then, by Lemma 2, we have that every point in F_v either lies within the convex hull of the v_i or in one of the spherical caps S^*_i for some i.

Since all the v_i are within R of $\pi(w_1)$, it follows that the convex hull of the v_i are inside $C^R_{\pi(w_1)}$. Lemma 4 says that S^*_i is contained in the spherical cap S'_i defined in terms of the ball $C^R_{\pi(w_1)}$ and the hyperplane $H_{\{\sim i\}}$, which proves that $S^*_i \subseteq C^R_{\pi(w_1)}$ for all i. Thus, it follows that $F_v \subseteq C^R_{\pi(w_1)}$.

□

6 Conclusions

This work shows that every signed graph on n nodes has a valid drawing in $\mathbb{R}^{n-2}$, and that there exists a signed graph on n nodes that does not have a valid drawing in $\mathbb{R}^{n-3}$. This demonstrates conclusively that $L(n) = n - 2$. Hence, different metric spaces or a *relaxed* definition of a valid drawing are needed to embed any graph on n nodes into $\mathbb{R}^k$ for $k < n - 2$.

The bound shown in this work depends only on the number of nodes n in the signed graph. However, it may be possible to show a tighter upper bound that also depends on a different parameter of the signed graph, such as the number of positive edges, the number of negative edges, or the ratio between positive and negative edges. Finally, no algorithm is known yet that finds, given a signed graph G, a small value k such that G has a valid drawing in $\mathbb{R}^k$. Such an algorithm would be an approximation algorithm (assuming P $\neq$ NP), since finding the smallest k is an NP-Hard problem.

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